

# Three-state Potts model in combination with the rock-scissors-paper game

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We study a three-state Potts model extended by allowing cyclic dominance between the states as it appears for the rock-scissors-paper game. Monte Carlo simulations are performed on a square lattice when varying the temperature and the strength of cyclic dominance. It is shown that the critical phase transition from the disordered state to the ordered one is destroyed by the cyclic dominance that yields a self-organizing pattern even at low temperatures. The differences and similarities are discussed between the present model and the half-filled, driven lattice gases with repulsive interaction.

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The ordering phenomena and the related phase transitions are already well understood in the equilibrium systems [1] while the theoretical understanding of the non-equilibrium phase transitions is still at its beginning [2, 3]. Many relevant and general features of these transitions in the equilibrium systems can be studied by the Potts models [4]. Now we introduce an extended version of the three-state Potts model to investigate the effect of cyclic dominance between the states. This model can be considered as a combination of the traditional Potts model and a spatial rock-scissors-paper game (sometimes called as three-state cyclic predator-prey or Lotka-Volterra models). The strength of cyclic dominance will be characterized by a single parameter ( $\varepsilon$ ) in such a way that for  $\varepsilon = 0$  the system becomes equivalent to the equilibrium Potts model exhibiting a well-known critical transition. The consideration of this model was strongly motivated by the work of Katz et al. [5, 6] who introduced the concept of driven lattice gases to study the effect of an external electric field on the ordering process. Since the appearance of their pioneering works many general features of these systems have already been explored (for a review see the Refs. [2, 7]).

The present model provides a continuous transition between an equilibrium system and a spatial evolutionary game where the time-reversal symmetry (detailed balance) is broken at the elementary (microscopic) steps. It will be shown that in the presence of cyclic dominance the long-range order cannot be observed even for low temperatures, and thereby the critical transition is also suppressed. A similar phenomenon has already been observed for the two-dimensional driven lattice gas with repulsive interactions where the formation of long-range order is prevented by an interfacial instability due to the enhanced particle transport along the boundaries separating the "chessboard" and "anti-chessboard" ordered phases [8, 9].

Our analysis is focused on a two-dimensional system where each site  $x$  of a square lattice is characterized by a three-state site variable, namely,  $s(x) = s_0, s_1$ , and  $s_2$ . For later convenience these states (strategies, species, etc.) will be denoted by the basis vectors of a three-dimensional space, i.e.

$$s_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (1)$$

The time evolution of the system is governed by random sequential updates. More precisely, the transition probability from a state  $s(x)$  (at site  $x$ ) to a randomly chosen state  $s'(x)$  is given as

$$W[s(x) \rightarrow s'(x)] = \frac{1}{1 + \exp(-\delta U(x)/T)} \quad (2)$$

where  $\delta U(x)$  is the difference of payoffs between the final and initial states, and  $T$  is the temperature characterising the effect of the noise. The payoff at site  $x$  depends on  $s(x)$  as well as on the neighboring states [ $s(y)$ ] as given by the following sum of matrix products:

$$U(x) = \sum_{\langle y \rangle} s^+(x) A s(y) \quad (3)$$

where the summation runs over the nearest-neighbors of the site  $x$ ,  $s^+(x)$  denotes the transpose of  $s(x)$ , and the payoff matrix  $A$  is defined as

$$A = \begin{pmatrix} 1 & \varepsilon & -\varepsilon \\ -\varepsilon & 1 & \varepsilon \\ \varepsilon & -\varepsilon & 1 \end{pmatrix}. \quad (4)$$

In the limit  $\varepsilon \rightarrow 0$  this model can be considered as a (kinetic) three-state ferromagnetic Potts model [4] with a Glauber dynamics [10]. Evidently, in this case the total energy is defined as  $H = -\sum_x U(x)/2$  and the microscopic processes satisfy the detailed balance in equilibrium. Consequently, the system tends towards a stationary state whose statistical features are described by the Gibbs ensemble. When decreasing the temperature the Potts model undergoes an ordering process from the disordered state to one of the three equivalent homogeneous (ordered) states. The corresponding critical transition represents a well-known universality class [1, 4].

For  $\varepsilon > 0$  the off-diagonal components of the payoff matrix  $A$  are asymmetrical therefore the total payoff (or the above defined  $H$ ) is not affected by the value of  $\varepsilon$  for any states. At the same time, the value of  $\varepsilon$  influences the probability of strategy changes because  $W[s(x) \rightarrow s'(x)]$  depends definitely on the variation of individual payoff [ $\delta U(x)$ ] and the above evolutionary rule manifests a way how the (selfish) individuals

wish to maximize their own payoff without any concern about the neighbors' performance. As a result, cyclic invasions occur along the boundaries separating homogeneous domains; domains of state  $s_0$  are invaded by  $s_1$  invaded by  $s_2$  invaded by  $s_0$ . These cyclic invasions are capable to maintain a self-organizing pattern with rotating spiral arms whose "velocity" is controlled by  $\varepsilon$  [11, 12, 13].

For most of the spatial evolutionary games the choice of the dynamical rules (or  $W[s(x) \rightarrow s'(x)]$ ) is based on a learning mechanism or strategy adoption modelling the Darwinian selections [14, 15, 16]. In these models different ways are suggested for the players to adopt the strategy of their more successful neighbors. The common feature of these strategy adoptions is that the new state will be equivalent to one of the neighboring strategies. Consequently, this mechanism prohibits the variation inside the homogeneous domains and makes the extinction process to be similar to those defined by the contact process (or directed percolation) [2, 3]. In the present model, however, the above "Glauber dynamics" allows the players to choose all the possible strategies therefore the time variation is not restricted to the interfaces separating the homogeneous domains. In the context of evolutionary game theory the above evolutionary rule describes a different behavior. Namely, here the players know all their possibilities and their choices depend on the increase of income what they are able to evaluate in the knowledge of neighboring strategies.

In the present work we study the effect of the "cyclic dominance" on the phase transition. For this purpose systematic Monte Carlo (MC) simulations are performed on a square lattice under periodic boundary conditions varying the temperature  $T$  and strength  $\varepsilon$  of the cyclic dominance for different linear sizes  $L$ . Each simulation is started from a random initial state and after a suitable thermalization time we have recorded the concentration of states ( $\rho_0$ ,  $\rho_1$ , and  $\rho_2$ ) for each Monte Carlo steps. We have also made simulations starting from ordered homogeneous phases to check the stability of the stationary state. To investigate the ordering process, we have determined the average value of the order parameter from the values of concentration data [4, 17]

$$m = \frac{1}{2} \langle [3 \max(\rho_0, \rho_1, \rho_2) - 1] \rangle \quad (5)$$

where  $\langle \dots \rangle$  refers to averaging over a sampling time varied from  $10^5$  to  $10^6$  Monte Carlo steps per sites (MCS). In the disordered phases  $\rho_0 = \rho_1 = \rho_2 = 1/3$  (due to the cyclic symmetry) and  $m = 0$  in the thermodynamic limit ( $L \rightarrow \infty$ ). For  $\varepsilon = 0$  and below the critical temperature ( $T < T_c = 0.995(1)$ ) the system evolves into one of the long-range ordered (symmetry breaking) stationary states (e.g.,  $\langle \rho_0 \rangle = (1 + 2m)/3$  and  $\langle \rho_1 \rangle = \langle \rho_2 \rangle = (1 - m)/3$  and the remaining two equivalent states are given by the cyclic permutation of indices) if the linear size is sufficiently large. In the thermodynamic limit the order parameter  $m$  decreases monotonously from 1 to 0 as the temperature is increased from 0 to  $T_c$  and the vanishing of  $m$  follows a power law behavior if  $T_c$  is approached from below [4]. For finite sizes, however, the MC simulations exhibit a smoothed order parameter func-

tion that deviates monotonously if we decrease the system size [17, 18]. Significantly different finite size effects are observed when investigating the present model for  $\varepsilon > 0$ .

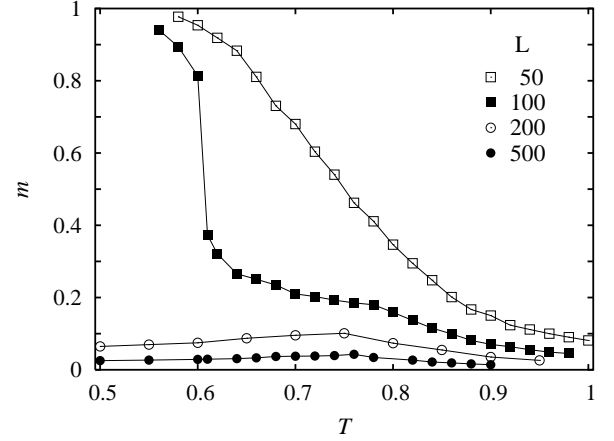


FIG. 1: Monte Carlo data for the order parameter vs. temperature at  $\varepsilon = 0.1$  for different system sizes as indicated. The solid lines are guides to the eye.

Figure 1 illustrates how the order parameter  $m$  varies with temperature  $T$  for different linear sizes if  $\varepsilon = 0.1$ . Apparently the MC data refers to an ordering process for small sizes ( $L = 50$  and  $100$ ) bearing a resemblance to MC data obtained for  $\varepsilon = 0$ . On the contrary, for example when  $L = 500$ , the MC data do not indicate the appearance of long-range order. Instead of it a self-organizing, three-color domain structure can be observed when visualizing the time-dependence of the spatial distribution (for a snapshot, see Fig. 2).

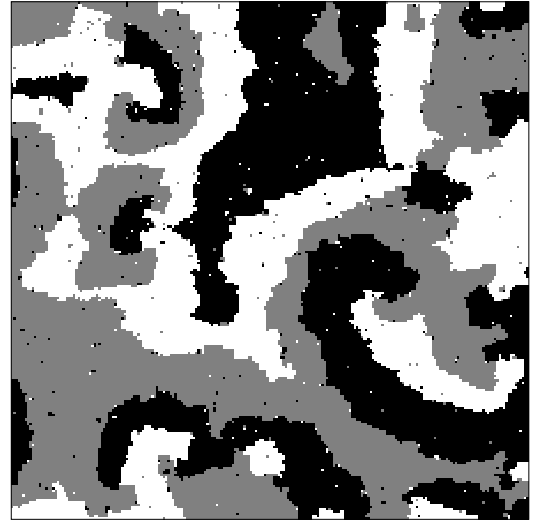


FIG. 2: Snapshot on a typical domain structure appearing for  $T = 0.64$  and  $\varepsilon = 0.1$ . The three-edge vortices (antivortices) rotate in clockwise (anticlockwise) direction with spiral arms because the average velocity of the invasion fronts (white invades black invades gray invades white) is hardly affected by their curvature.

On this snapshot one can identify all the three ordered

phases forming domains with a characteristic linear size  $l$ . For  $T < T_c$  and  $\varepsilon = 0$  the growth of these domains ( $l \sim \sqrt{t}$ ) is driven by the interfacial (Potts) energy [19, 20, 21]. Here, however, this domain growth is prevented by two processes emerging for  $\varepsilon > 0$ . The first process is related to the appearance of rotating spiral arms for the three-edge vortices where the three types of domains (or domain boundaries) meet. On these maps we can distinguish vortices and antivortices rotating in opposite directions. Some topological and geometrical features of such spatio-temporal patterns were already investigated in previous papers [11, 12, 13, 22]. It is found that the spirals become well-marked if a “surface tension” is switched on and then the corresponding patterns cannot be characterized by a single parameter (e.g., typical domain size or correlation length) [22].

In the present model there exists a second process causing the appearance of growing domains via a nucleation mechanism inside the large “homogeneous” territories. Due to this process an “ordered state” prevailed by  $s_0$  will be transformed into another one prevailed by  $s_1$  as indicated by simulations for small sizes. In these cases the three ordered states follow each other cyclically and the above method yields a sufficiently large value for  $m$  (see Fig. 1). Both the duration time and the probability of these transitions, that are initiated by a nucleation mechanism due to the thermal fluctuations, increase with the system size. This is the reason why the values of  $m$  are higher for  $L = 50$  than those for  $L = 100$  in the Fig. 1.

For sufficiently large system sizes ( $L \gg l$ ) both mentioned mechanisms work simultaneously and result in a self-organizing pattern where the three states are present with the same concentration and  $m = 0$ ). Henceforth the quantitative investigations will be focused on the large systems ( $L > 500$ ) and to the region of temperature ( $T > 0.6T_c$ ) where the spatial patterns are isotropic.

Now we study the variation of correlation length  $\xi$  derived from the asymptotic behavior (exponential vanishing) of the two-site correlation function [8]. For this purpose a series of MC simulations is performed by varying the temperature for  $\varepsilon = 0.1$ . The inset in Fig. 3 illustrates the absence of divergence in  $\xi$  as expected. When decreasing the temperature the correlation length increases monotonously until a maximum value ( $\xi \simeq 11.5(5)$  measured in lattice unit). Below the peak at  $T \simeq 0.77$  the visualization of the distribution of species shows a self-organizing pattern (see Fig. 2) and here the value of  $\xi$  decreases very slowly with  $T$ .

The  $\varepsilon$ -dependence of the correlation length is also determined for a fixed temperature and the results are illustrated in a log-log plot (see Fig. 3). These MC data are consistent with a prediction  $\xi \sim 1/\varepsilon$  for small  $\varepsilon$ . In the typical size of domains similar divergence was found previously for a model where the nucleation mechanism was blocked [22]. This observation refers to a minor role of nucleation mechanism in the maintenance of the self-organizing patterns at sufficiently low temperature ( $T < T_c$ ). For higher temperature the nucleation mechanism plays a crucial role by preventing the formation of monodomain state even for  $\varepsilon = 0$ . Thus it is conjectured that this process results in a different behavior of  $\xi$  at the critical

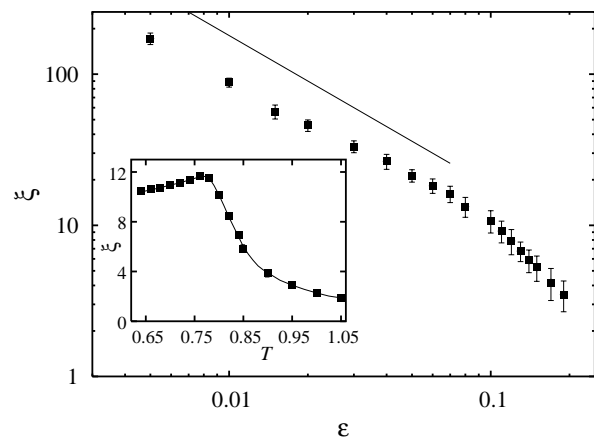


FIG. 3: Variation of correlation length (with error bars) as a function of  $\varepsilon$  for fixed temperature ( $T = 0.75$ ). The straight line indicates a power law divergence with a slope of  $-1$ . Inset shows the correlation length vs. temperature for  $\varepsilon = 0.1$ .

temperature. Furthermore, here it is worth mentioning that in the above mentioned driven lattice gas model the transversal correlation length was also proportional to the inverse of the strength of driving field [9].

In such systems the Potts energy measures the concentration of domain walls and it gives an additional information about spatial distributions. From the average Potts energy as a function of temperature one can derive a specific heat ( $c = d\langle H \rangle / dT$ ) that exhibits a  $\lambda$ -divergence at the critical temperature in the equilibrium limit ( $\varepsilon = 0$ ). Figure 4 illustrates how the  $\lambda$ -divergence is smoothed out if the cyclic dominance is switched on. When choosing larger and larger  $\varepsilon$  the maximum value of specific heat decreases meanwhile the peak position moves towards the lower and lower temperatures. The peaks are so shallow in the “driven” cases that a logarithmic scale was necessary to present them in the same figure. The appearance of this peak in the specific heat can be interpreted as a sign of the short range ordering.

A similar phenomenon was observed for the driven lattice gases with repulsive interaction when increasing the external electric field [8, 9]. Although the observed patterns and microscopic mechanisms are very different, in both cases interfacial effects prevent the formation of long-range order in the presence of driving force. In these cases the interfaces belong to the stationary states and their geometrical characterization requires additional parameters. This general feature can occur in some other non-equilibrium systems (e.g., in ecological models) where an external force induces some extra activity along the interfaces separating the “ordered domains”.

In summary, a three-state dynamical lattice model is introduced by combining the Potts model and the rock-scissors-paper game to study the effect of cyclic dominance on the ordering process. Due to the cyclic dominance the time-reversal symmetry is broken at the elementary steps and thereby the behavior of this model cannot be described by the methods of equilibrium statistical physics. Our numerical analyses have justified that both the long-range (symmetry breaking)

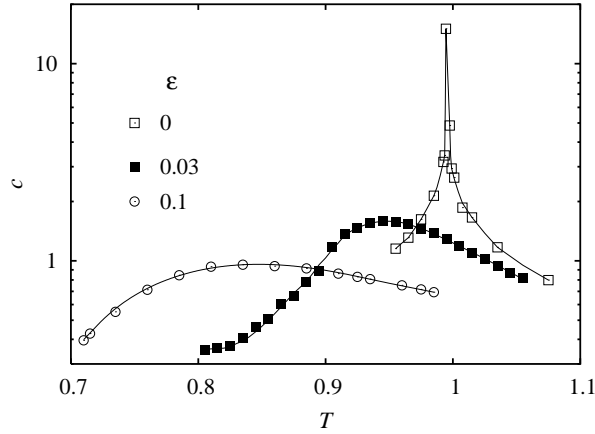


FIG. 4: Specific heat as a function of temperature for three different values of  $\varepsilon$  as indicated. These MC data are obtained for such a large linear size ( $L = 1000$ ) where the size effects are already negligible.

ordering process and the corresponding critical transition are suppressed in the presence of cyclic dominance ( $\varepsilon > 0$ ). According to the simulations the three equivalent ordered phases coexist by forming a self-organizing domain structure even at low temperatures and sufficiently weak cyclic dominance. The equilibrium state is approached via the divergence of the typical domain size when the strength of cyclic dominance goes to zero.

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